Robustness and nondegenerateness for linear complementarity problems

Jansen, M.J.M.; Tijs, S.H.

Published in:
Mathematical Programming

Publication date:
1987

Mathematical Programming 37 (1987) 293-308
North-Holland

ROBUSTNESS AND NONDEGENERATENESS FOR LINEAR COMPLEMENTARITY PROBLEMS

M.J.M. JANSEN
Open University, Heerlen, The Netherlands

S.H. TIJS
Department of Mathematics, University of Nijmegen, Nijmegen, The Netherlands

Received 10 June 1985
Revised manuscript received 11 April 1986

In this paper the main focus is on a stability concept for solutions of a linear complementarity problem. A solution of such a problem is robust if it is stable against slight perturbations of the data of the problem. Relations are investigated between the robustness, the nondegenerateness and the isolatedness of solutions. It turns out that an isolated nondegenerate solution is robust and also that a robust nondegenerate solution is isolated. Since the class of linear complementarity problems with only robust solutions or only nondegenerate solutions is not an open set, attention is paid to Garcia's class G_n of linear complementarity problems. The nondegenerate problems in G_n form an open set.

Key words: Linear complementarity problem, robustness, perturbation, classes of matrices.

1. Introduction

Several problems arising in mathematical programming, game theory, economics and engineering can be translated into the following form (cf. [3]):

Find, for a given $n \times n$-matrix M and an n-vector q, vectors $z \in \mathbb{R}^n$, satisfying

$$z \geq 0, \tag{1.1}$$

$$w := Mz + q \geq 0, \tag{1.2}$$

$$\langle w, z \rangle = w^{\mathrm{t}} z = 0. \tag{1.3}$$

Hence, the problem, specified by (1.1)-(1.3), is to find nonnegative vectors z such that the image w of z under the affine map $z \mapsto Mz + q$ is nonnegative and perpendicular to z. This problem is known as the *linear complementarity problem* (LCP) corresponding to q and M and will be denoted by (q, M). The number n will be called the *size* of the problem (q, M). A vector z, satisfying (1.1) and (1.2), is called *feasible* and a feasible vector satisfying also (1.3) is called a *solution* of the linear complementarity problem (q, M). The set of all such solutions will be denoted by $S(q, M)$.

A majority of papers in the LCP-field have their main focus on existence of a solution, on finding one such solution for a given linear complementarity problem or demonstrating that such a solution does not exist. In many practical situations, however, which translate into a linear complementarity problem, one will not be indifferent with respect to the various solutions, if there are more than one. If there can be made a choice between solutions one can consider which solutions have additional interesting properties. In this study we will concentrate on three such properties: robustness, isolatedness and nondegenerateness.

Here we call a solution of a linear complementarity problem robust if it is stable against slight perturbations of the data of the problem. In [14], Robinson gives conditions for a solution under which each slightly disturbed problem has a unique solution near the original one. His results are based on the theory of (strong regularity of) generalized equations. The same ideas are used in a paper of Ha [6], who introduced two stability concepts. The difference and similarity between his concepts of stability and ours will be discussed in this paper.

Of much help investigating the relations between the three properties will be the paper of Jansen [7], in which the structure of the solution set of linear complementarity problems is extensively studied. The main results of that paper are summarized in Section 2. Section 3 deals with nondegenerate and isolated solutions while in Section 4 robust solutions are central. In Section 5, for a subclasss of linear complementarity problems, interesting stability properties are obtained.

We conclude this section with some notational remarks. For a natural number n, we denote the set $\{1, 2, \ldots, n\}$ by $\mathbb{N}_n$. The elements of the standard basis of unit vectors of $\mathbb{R}^n$ are denoted by $e_1, \ldots, e_n$. For a vector $z \in \mathbb{R}^n$, the *carrier* $\{i \in \mathbb{N}_n : z_i \neq 0\}$ is denoted by $C(z)$ and $\|z\|_\infty := \max\{|z_i|: i \in \mathbb{N}_n\}$.

For a vector $z \in \mathbb{R}^n$, the ε-ball $\{z' \in \mathbb{R}^n : \|z - z'\|_\infty < \varepsilon\}$ around z is denoted by $B_\varepsilon(z)$. For a set $S \subset \mathbb{R}^n$, $\operatorname{conv}(S)$ is the convex hull of S and $B_\varepsilon(S) := \{z \in \mathbb{R}^n$: there is an $s \in S$ such that $\|z - s\|_\infty < \varepsilon\}$. If $C \subset \mathbb{R}^n$ is a convex set, then we write relint (C) for the relative interior of C. $\mathbb{R}^n_+$ is the positive orthant in $\mathbb{R}^n$, consisting of vectors with only nonnegative coordinates. For a finite set X, the number of elements is denoted by $|X|$. The family of real $n \times n$-matrices is denoted by $\mathbb{R}^{n \times n}$.

2. Preliminaries

In this section we recall some well-known facts. Let (q, M) be a linear complementarity problem of size n. Then we have:

$$z \in S(q, M) \text{ iff } z \text{ is feasible and } C(z) \subset E(z) \tag{2.1}$$

where $E(z) := \{j \in \mathbb{N}_n : e_j^t Mz + q_j = 0\}$. In the paper of Jansen [7], concerning the structure of the solution set of LCP's, a main role is played by convex components, defined as follows.

Definition 2.1. A non-empty convex subset P of $S(q, M)$ is called a *convex component* of $S(q, M)$, if there exists a no convex subset of $S(q, M)$, properly containing P.

In [7] it is shown that for an LCP with non-empty solution set, this solution set is a finite union of convex components and that each convex component is a polyhedral set. The following characterization of *extreme solutions*, i.e. extreme points of convex components, can also be found in [7].

Theorem 2.2. *Let z be a solution unequal to zero of the linear complementarity problem (q, M). Then the following assertions are equivalent:*

(i) *z is an extreme solution*

(ii) *there exists a nonsingular square submatrix K of M such that $z_j = 0$ if the j-th column of M plays no role in K and such that $z_K = -K^{-1}q_K$, where z_K (q_K) is the vector obtained from z (q) by removing the coordinates corresponding to the columns (rows) of M which play no role in K.*

For a convex component P of $S(q, M)$ we denote the carrier $\bigcup_{z\in P} C(z)$ of P by $C(P)$ and the set $\bigcap_{z\in P} E(z)$ by $E(P)$. Note that $\mathbb{N}_n - E(P) = \bigcup_{z\in P} C(Mz+q)$, so $\mathbb{N}_n - E(P)$ is the carrier of the image of P under the map $z \mapsto Mz+q$.

In the next lemma P and relint(P) are described with the aid of $C(P)$ and $E(P)$.

Lemma 2.3. *Let P be a convex component of $S(q, M)$. Then*

(i) $C(P) \subset E(P)$,

(ii) $P = \{z \in \mathbb{R}^n$: *z is feasible*, $C(z) \subset C(P)$, $E(z) \supset E(P)\}$,

(iii) $\text{relint}(P) = \{z \in \mathbb{R}^n$: *$z$ is feasible*, $C(z) = C(P)$, $E(z) = E(P)\}$.

Proof. (i) First we observe that for feasible points z' and z'' and for each z of the form $z = \alpha z' + (1-\alpha)z''$ with $0 < \alpha < 1$ we have

$$C(z) = C(z') \cup C(z''), \qquad E(z) = E(z') \cap E(z''). \tag{2.2}$$

If z' and z'' belong to the convex component P, then also such a $z \in P$, which implies that $C(z) \subset E(z)$. Therefore $C(z') \subset E(z'')$ for each $z', z'' \in P$. So $C(P) \subset E(P)$.

(ii) Let $P^* := \{z \in \mathbb{R}^n$: z is feasible, $C(z) \subset C(P)$, $E(z) \supset E(P)\}$. We prove that $P = P^*$. Obviously $P \subset P^*$. Now let $z' \in P^*$. We have to show that $z' \in P$. By (i) and the definition of P^* we have $C(z') \subset E(z')$, so $z' \in S(q, M)$. Let z'' be an arbitrary point in P and let $z = \alpha z' + (1-\alpha)z''$ where $\alpha \in (0, 1)$. By (2.2) and the definition of P^*, z is feasible and $C(z) \subset E(z)$. Therefore $z \in S(q, M)$. This implies the conv($P \cup \{z'\}$) is a convex subset of $S(q, M)$ containing the convex component P. But then conv($P \cup \{z'\}) = P$, and $z' \in P$.

(iii) Let $P^{**} := \{z \in \mathbb{R}^n$: z is feasible, $C(z) = C(P)$, $E(z) = E(P)\}$. We have to prove that $P^{**} = \text{relint}(P)$. If $|P| = 1$ then this is true. Suppose $|P| > 1$. Take $z' \in \text{relint}(P)$. For each $z'' \in P$ there is a $\mu > 1$ such that $z = (1-\mu)z'' + \mu z' \in P$ (Theorem 6.4, Rockafellar [15]). In other words z' is a convex combination of z and z''. By

(2.2), $C(z') \supset C(z'')$, $E(z') \subset E(z'')$. Since z'' is arbitrary in P, $C(z') = C(P)$ and $E(z') = E(P)$. Hence $z' \in P^{**}$. So we have shown that $\mathrm{relint}(P) \subset P^{**}$. Conversely, let $z' \in P^{**}$. Let z'' be an arbitrary point in P. Then

$$C(z'') \subset C(P) = C(z') \quad \text{and} \quad E(z'') \supset E(P) = E(z'). \tag{2.3}$$

Since $C(Mz'+q)$ is the complement of $E(z')$ in $\mathbb{N}_n$ we have

$$C(Mz''+q) \subset C(Mz'+q). \tag{2.4}$$

In view of (2.3) and (2.4) there is a $\bar{\mu} > 1$ such that for all μ with $1 < \mu \leqslant \bar{\mu}$: $z = (1-\mu)z'' + \mu z'$ is feasible. Since z' is a convex combination of z and z'', by (2.2) and (i) we obtain $C(z) \subset C(z') = C(P) \subset E(P) = E(z') \subset E(z)$. Hence $z \in P$. By Theorem 6.4 in Rockafellar [15], $z' \in \mathrm{relint}(P)$. So $P^{**} \subset \mathrm{relint}(P)$. □

In the next theorem, which is proved in [7], a relation is given between the dimension of (the affine hull of) a convex component P of $S(q, M)$ and the rank of the submatrix

$$M(P) := [m_{ij}]_{i \in E(P), j \in C(P)}$$

of M.

Theorem 2.4. *Let (q, M) be a linear complementarity problem and let $P \neq \{0\}$ be a convex component of $S(q, M)$. Then* $\dim(P) = |C(P)| - \mathrm{rank}(M(P))$.

Theorem 2.4 can be used to give an elementary proof of the well known result (cf. Theorem 3.2 of [11]) that a linear complementarity problem (q, M) has a finite number of solutions if M has only nonsingular principal submatrices: Consider for a convex component P of $S(q, M)$ the principal submatrix $N(P) := [m_{ij}]_{i \in E(P), j \in E(P)}$ of M. Since $N(P)$ is nonsingular, the rank of the submatrix $M(P)$ of $N(P)$ equals $|C(P)|$. Hence Theorem 2.4 implies that $\dim(P) = 0$; so P is a one-point-set.

3. Nondegenerate solutions and isolated solutions

As already noted, for a solution z of a linear complementarity problem (q, M), in general, $C(z)$ is a subset of $E(z)$. Solutions where these sets coincide deserve some special attention. Those solutions are considered in this section, together with isolated solutions.

Definition 3.1 (cf. Mangasarian [11, 12]). A solution z of the problem (q, M) is called *nondegenerate* if the two equivalent assertions hold:

(i) $C(z) = E(z)$,

(ii) $q + Mz + z > 0$.

The problem (q, M) is called a *nondegenerate problem* if the solution set is nonempty and all solutions are nondegenerate.

The next theorem states that a nondegenerate solution cannot lie on the relative boundary of a convex component, if the component consists of more than one point. Note that for a one-point-set S, $\text{relint}(S) = S$.

Theorem 3.2. *Let P be a convex component of the solution set of the linear complementarity problem (q, M) and let $z \in P$. If z is nondegenerate, then $z \in \text{relint}(P)$.*

Proof. For each $z \in P$ we have, in view of Lemma 2.3(i),

$$C(z) \subset C(P) \subset E(P) \subset E(z). \tag{3.1}$$

If z is nondegenerate all inclusions in (3.1) can be replaced by equalities, and then Lemma 2.3(iii) implies that $z \in \text{relint}(P)$. □

It there are no degenerate solutions by Theorem 3.2 the convex components consist of one point only. As noted in Section 2, there are only a finite number of convex components. Consequently, a nondegenerate linear complementarity problem has only a finite number of solutions, which is well known.

The next example shows that a solution of (q, M), which is in the relative interior of a convex component, is not necessarily nondegenerate. It also shows that not all linear complementarity problems with a non-empty solution set also have a nondegenerate solution.

Example 3.3. Let

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad q = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Then $S(q, M) = \{z \in \mathbb{R}^2_+ : z_1 = 0\}$. Hence, all solutions are degenerate.

Definition 3.4. A solution z of a linear complementarity problem (q, M) is called *isolated* (or *locally unique*, cf. [11]) if $\{z\}$ is a convex component of the solution set $S(q, M)$.

With the aid of Theorems 2.4 and 3.2 we can easily prove the following result of Mangasarian, characterizing nondegenerate solutions unequal to zero, which are isolated.

Lemma 3.5 (cf. [11, Corollary 3.2]). *Let z be a nondegenerate solution, unequal to zero, of the problem (q, M). Then z is isolated if and only if the submatrix $[m_{ij}]_{i \in C(z), j \in C(z)}$ of M has full rank.*

Proof. If z is isolated and nondegenerate, then for $P = \{z\}$ we have $C(z) = C(P) = E(P) = E(z)$. Hence, the submatrix in the theorem coincides with $M(P)$ and by Theorem 2.4, $0 = \dim(P) = |C(z)| - \text{rank}(M(P))$, so $M(P)$ has full rank. Conversely, if Z is nondegenerate and the submatrix in the theorem has full rank, then for each convex component P containing z, we have $\dim(P) = |C(P)| - \text{rank}(M(P)) = 0$. So $P = \{z\}$, z is isolated. □

With a linear complementarity problem (q, M) of size n we associate the infinitely often differentiable mapping $f:\mathbb{R}^n \to \mathbb{R}^n$ defined by

$$f_i(z) := z_i(e_i^t Mz + q_i) \quad \text{for } i \in \mathbb{N}_n.$$

Note that z is a solution of (q, M) if and only if z is feasible and $f(z) = 0$.

Calling a solution *regular* if the Jacobian $J_f(z) := [\partial_j f_i(z)]_{i,j=1}^n$ of f evaluated at z is nonsingular, we have the following

Lemma 3.6. *A solution of a linear complementarity problem is isolated and nondegenerate if and only if it is a regular one.*

Proof. Since

$$\partial_j f_i(z) = \begin{cases} z_i m_{ij}, & j \neq i, \\ z_i m_{ii} + e_i^t Mz + q_i, & j = i, \end{cases}$$

one can show that, for a solution z,

$$\det J_f(z) = \prod_i q_i \quad \text{if } z = 0$$

and, for $z \neq 0$,

$$|\det J_f(z)| = \prod_{i \in C(z)} z_i \prod_{i \notin C(z)} (e_i^t Mz + q_i) |\det[m_{ij}]_{i \in C(z), j \in C(z)}|.$$

Note that by definition a product over the empty set is equal to one. Now for $z = 0$ the lemma follows from the fact that the nondegenerateness of 0 is equivalent with $q > 0$. For $z \neq 0$ the lemma is an easy consequence of Lemma 3.5. □

For a linear complementarity problem (q, M) of size n the matrix M together with vector q can be identified with elements of $\mathbb{R}^n \times \mathbb{R}^{n \times n}$, whose set we provide with a standard metric d, where for $(q, M), (q', M') \in \mathbb{R}^n \times \mathbb{R}^{n \times n}$,

$$d((q, M), (q', M')) := \max\{\|q - q'\|_\infty, \|M - M'\|_\infty\}.$$

Here $\|A\|_\infty := \max\{|a_{ij}|: i \in \mathbb{N}_n, j \in \mathbb{N}_n\}$ for $A = [a_{ij}]_{i=1,}^n{}_{j=1}^n \in \mathbb{R}^{n \times n}$.

Lemma 3.7. *Let* $(q^1, M^1), (q^2, M^2), \ldots$ *be a sequence in* $\mathbb{R}^n \times \mathbb{R}^{n \times n}$ *converging to* (q, M). *Let* $z^1, z^2, \ldots$ *be a sequence in* $\mathbb{R}^n$ *converging to* z *such that* $z^k \in S(q^k, M^k)$ *for all* $k \in \mathbb{N}$. *Then*

(i) *there exists a* $K \in \mathbb{N}$ *such that, for all* $k \geq K$,

$$C(z) \subset C(z^k), \; E(z) \supset E(z^k).$$

(ii) $z \in S(q, M)$.

(iii) *if* z *is a nondegenerate solution of* (q, M), *then* z^k *is a nondegenerate solution of* (q^k, M^k) *for large* k.

(iv) *if* z *is a nondegenerate and isolated solution of* (q, M), *then* z^k *is a nondegenerate and isolated solution of* (q^k, M^k) *for large* k.

Proof. Because

$$\lim_{k\to\infty} z^k = z \quad \text{and} \quad \lim_{k\to\infty} M^k z^k + q^k = Mz + q \tag{3.2}$$

there is a $K \in \mathbb{N}$ such that $C(z^k) \supset C(z)$ and $C(M^k z^k + q^k) \supset C(Mz+q)$ for all $k \geq K$ and this implies (i). Since (3.2) implies that z is feasible for (q, M) and since $C(z) \subset C(z^K) \subset E(z^K) \subset E(z)$ by (i), it follows from (2.1) that $z \in S(q, M)$, proving (ii).

If z is a nondegenerate solution of (q, M), then, by (i)

$$C(z^k) \supset C(z) = E(z) \supset E(z^k) \supset C(z^k) \quad \text{for } k \geq K \tag{3.3}$$

implying that z^k is a nondegenerate solution of (q^k, M^k) if $k \geq K$. Part (iv) is an easy consequence of Theorem 3.8. □

From Lemma 3.7(ii) we conclude immediately that the multifunction $S: \mathbb{R}^n \times \mathbb{R}^{n\times n} \twoheadrightarrow \mathbb{R}^n$, assigning to $(q, M) \in \mathbb{R}^n \times \mathbb{R}^{n\times n}$ the solution set $S(q, M)$ of the linear complementarity problem (q, M), has a closed graph.

In the next theorem we show that in a neighbourhood of a point $(\bar{q}, \bar{M}, \bar{z})$, where $\bar{z}$ is an isolated and nondegenerate solution of the problem $(\bar{q}, \bar{M})$, the set $\{(q, M, z): z \in S(q, M)\}$ is a smooth curve through the point $(\bar{q}, \bar{M}, \bar{z})$.

Theorem 3.8. *Let $\bar{z}$ be an isolated and nondegenerate solution of a linear complementarity problem $(\bar{q}, \bar{M})$ of size n. Then there exist neighbourhoods U of $(\bar{q}, \bar{M})$ and V of $\bar{z}$ such that*

(i) $|S(q, M) \cap V| = 1$, *for all* $(q, M) \in U$, *and*

(ii) *the mapping* $\sigma: U \to V$ *defined by* $\{\sigma(q, M)\} = S(q, M) \cap V$ *is differentiable.*

Proof. Let, for $i \in \mathbb{N}_n$,

$$F_i(q, M, z) := z_i(e_i^t Mz + q_i).$$

Then $F(q, M, z) = 0$ if $z \in S(q, M)$ and, in view of Lemma 3.6, z is an isolated and nondegenerate solution of (q, M) if and only if the determinant of the matrix

$$\left[\frac{\partial F_i(q, M, z)}{\partial z_j}\right]_{i,j=1}^{n}$$

is unequal to zero.

By applying the implicit Function Theorem to the mapping F, we can find neighbourhoods U of $(\bar{q}, \bar{M})$ and V of $\bar{z}$ and a unique differentiable mapping $\sigma: U \to V$ such that

$$\{(q, M, z) \in U \times V: F(q, M, z) = 0\} = \{(q, M, \sigma(q, M)): (q, M) \in U\}.$$

By choosing U and V sufficiently small, since z is a nondegenerate solution, we can arrange that $z \in S(q, M)$ if $(q, M, z) \in U \times V$ and $F(q, M, z) = 0$. Since every solution $z \in V$ of $(q, M) \in U$ satisfies $F(q, m, z) = 0$, z is the only solution of (q, M) which is in V. □

We conclude this section with some remarks.

Remark 3.9. The family $\mathrm{SOL}_n := \{(q, M) \in \mathbb{R}^n \times \mathbb{R}^{n\times n}: S(q, M) \neq \emptyset\}$ of linear complementarity problems of size n with nonempty solution set is neither an open set nor a dense set in $\mathbb{R}^n \times \mathbb{R}^{n\times n}$. That SOL_n is not open follows for instance by noting that $(0, -I_n) \in \mathrm{SOL}_n$ and, for each $\varepsilon > 0$, $(-\varepsilon 1_n, -I_n) \notin \mathrm{SOL}_n$, while $d((0, -I_n), (-\varepsilon 1_n, -I_n)) = \varepsilon$. Hence I_n is the $n\times n$-identity matrix and 1_n the vector in $\mathbb{R}^n$ with all coordinates equal to 1. That SOL_n is not dense follows from the fact that each linear complementarity problem (q, M) with $d((q, M), (-1_n, -J_n)) < 1$ has no solution, where $J_n = [1]_{i,j=1}^n$. SOL_n is also not closed: $(-1_n, \varepsilon I_n) \in \mathrm{SOL}_n$ for all $\varepsilon > 0$, but $(-1_n, 0 I_n) \notin \mathrm{SOL}_n$.

Remark 3.10. The family of nondegenerate linear complementarity problems of size n is not open for general n.

For $n = 3$ this follows from the observations that for

$$q = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad M_k = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -k^{-1} & 0 \\ 1 & k^{-1} & 0 \end{bmatrix}$$

we have:

(i) $S(q, M) = \{(1, 0, 1)\}$ and $(1, 0, 1)$ is nondegenerate,

(ii) $(0, k, 1)$ is a degenerate solution of (q, M_k),

(iii) $\lim_{k\to\infty}(q, M_k) = (q, M)$.

Remark 3.11. The family ONS_n of linear complementarity problems of size n with at least one nondegenerate solution is a dense subset of SOL_n. This can be seen as follows. Take $(q, M) \in \mathrm{SOL}_n$ and let z be a solution of (q, M). Then z is for each $\varepsilon > 0$ a nondegenerate solution of $(q + \varepsilon u, M)$, where $u = (u_1, u_2, \ldots, u_n)$ is such that $u_i = 1$ if $i \in E(z) - C(z)$ and $u_i = 0$ otherwise. Now $d((q, M), (q + \varepsilon u, M)) = \varepsilon$ and $(q + \varepsilon u, M) \in \mathrm{ONS}_n$.

4. Robust solutions

In this section we consider solutions, which are stable against slight perturbations of the data of the linear complementarity problem.

Definition 4.1. Let (q, M) be a linear complementarity problem. A solution z of (q, M) is called *robust* if for each neighbourhood V of z there exists a neighbourhood U of (q, M) such that $S(q', M') \cap V \neq \emptyset$, for all $(q', M') \in U$.

If $\mathrm{SROB}(q, M)$ denotes the set of all robust solutions of (q, M), then this linear complementarity problem is called a *robust problem* if $\mathrm{SROB}(q, M) = S(q, M) \neq \emptyset$.

For example the problem $(0, M)$ of Example 3.3 is a linear complementarity problem without a robust solution because $S(-\varepsilon 1_2, M) = \emptyset$ for each $\varepsilon > 0$.

Two other stability concepts were recently considered in Ha [6]. First of all he calls a solution z of a linear complementarity problem (q, M) *stable* if there are neighbourhoods V of z and U of (q, M) such that

(i) $S(q', M') \cap V \neq \emptyset$, for all $(q', M') \in U$, and

(ii) $\sup\{\|z' - z\|_\infty : z' \in S(q', M') \cap V\} \to 0$ as (q', M') approaches (q, M).

If these neighbourhoods U and V can be chosen in such a way that

(iii) $|S(q', M') \cap V| = 1$, for all $(q', M') \in U$,

then the solution z is called *strongly stable.*

In the next theorem the relationship between one of Ha's concepts of stability and ours is investigated.

Theorem 4.2. *A solution of a linear complementarity problem is stable if and only if it is isolated and robust.*

Proof. Let z be an isolated and robust solution of a linear complementarity problem (q, M). Since z is isolated, we can find a bounded neighbourhood V of z such that

$$S(q, M) \cap \bar{V} = \{z\}. \tag{4.1}$$

Since z is robust, there exists a neighbourhood U of (q, M) such that

$$S(q', M') \cap V = \emptyset, \text{ for all } (q', M') \in U.$$

Note that

$$\begin{aligned}\sup\{\|z - z'\|_\infty : z' \in S(q', M') \cap V\} &\leqslant \sup\{\|z - z'\|_\infty : z' \in \bar{V}\} \\ &\leqslant \max\{\|z'' - z'\|_\infty : z', z'' \in \bar{Z}\} =: d.\end{aligned}$$

So $\sup\{\|z - z'\|_\infty : z' \in S(q', M') \cap V\} \in [0, d]$, for all $(q', M') \in U$. Therefore, if condition (ii) is not satisfied, there exists a sequence $(q(1), M(1)), (q(2), M(2)), \ldots$ in U converging to (q, M) with

$$\lim_{n\to\infty} \sup\{\|z - z'\|_\infty : z' \in S(q(n), M(n)) \cap V\} = s \neq 0.$$

Choose for all $n \in \mathbb{N}$ a solution $z(n) \in S(q(n), M(n)) \cap V$ such that

$$\|z - z(n)\|_\infty > \sup\{\|z - z'\|_\infty : z' \in S(q(n), M(n)) \cap V\} - \frac{1}{n}.$$

Since $z(n) \in \bar{V}$ for all $n \in \mathbb{N}$ and $\bar{V}$ is bounded, we may suppose that the sequence $z(1), z(2), \ldots$ converges, say to z^*. In view of Lemma 3.7(ii), $z^* \in S(q, M) \cap \bar{V}$. Because $\|z - z^*\|_\infty \geqslant s \neq 0$, this contradicts (4.1). Hence z is a stable solution.

Since the other implication in the theorem is trivial, the proof is complete. □

Remark 4.3. A solution z is strongly stable if and only if the above condition (iii) is satisfied and

(iv) the mapping $\sigma: U \to V$ defined by $\{\sigma(q, M)\} = S(q, M) \cap V$ is continuous. Thus Theorem 3.8 implies that an isolated and nondegenerate solution is strongly stable (cf. Corollary 3.2 of Ha [6]).

Since a strongly stable solution is robust we also have the following

Lemma 4.4. *An isolated and nondegenerate solution of a linear complementarity problem is robust.*

In the next theorem we give sufficient conditions for the solution $z = 0$ to be robust.

Theorem 4.5. *Suppose that* 0 *is a nondegenerate solution of* (q, M). *Then* 0 *is a robust and isolated solution.*

Proof. The nondegenerateness of 0 is equivalent to $q > 0$. Then $0 \in S(q', M')$ if $\|q - q'\|_\infty < \delta := \min_{i \in \mathbb{N}_n} q_i$ and M' is arbitrary, which implies that 0 is robust. Let P be a convex component of $S(q, M)$, containing 0. Then

$$C(P) \subset E(P) \subset E(0) = C(0) = \emptyset \text{ implies that } P = \{0\}.$$

Hence 0 is an isolated solution. $\square$

Remark 4.6. The converse of Theorem 4.5 is not true. For

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad q = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

0 turns out to be a robust and isolated solution but 0 is degenerate. That 0 is degenerate and isolated is immediate. For $\varepsilon < 1$ take $\delta = \frac{1}{2}\varepsilon$. Then for (q', M') with $d((q, M), (q', M')) < \delta$ we have $0 \in S(q', M')$ if $q_2' \geq 0$ and $(0, -(m_{22}')^{-1} q_2') \in S(q, M)$ if $q_2' < 0$ and $-(m_{22}')^{-1} q_2' < (1 - \delta)^{-1}\delta \leq 2\delta = \varepsilon$. This implies that 0 is robust.

The following theorem also relates the concepts of robustness, nondegenerateness and isolatedness.

Theorem 4.7. *Let* $z \in \mathbb{R}^n$ *be a robust and nondegenerate solution of* (q, M). *Then* z *is an isolated solution.*

Proof. If $z = 0$, the result follows from Theorem 4.5. Hence, suppose $z \neq 0$ and w.l.o.g. that $C(z) = \{1, 2, \ldots, s\}$. Since z is nondegenerate, by Lemma 3.5, z is an isolated solution if and only if the submatrix $N := [m_{ij}]_{i \in C(z), j \in C(z)}$ is nonsingular. Suppose that z is not isolated. Then N is singular. Now consider the system of equations

$$Nx + q^* = 0,$$

where $x \in \mathbb{R}^s$ and $q^* := (q_1, q_2, \ldots, q_s)$.

For any $\varepsilon > 0$ there is a $p \in \mathbb{R}^s$ such that $\| p - q^* \|_\infty < \varepsilon$ and the system of equations $Nx + p = 0$ has no solution. Let $q' := (p_1, \ldots, p_s, q_{s+1}, \ldots, q_n)$. Then, by nondegenerateness of z and by Lemma 3.7(iii), (q', M) has no solution near z, which contradicts the robustness of z. □

Combining the foregoing results we obtain

(a) that a nondegenerate solution of a linear complementarity problem is robust and even strongly stable if and only if it is an isolated one, and

(b) that a nondegenerate linear complementarity problem is strongly stable (cf. Ha [6, Corollary 3.3]).

We have proved that robust, nondegenerate solutions are isolated and that isolated, nondegenerate solutions are robust. The following example and Remark 4.6 show that an isolated, robust solution is not necessarily nondegenerate.

Example 4.8. Let

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad q = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

Then $S(q, M) = \{e_1\}$ and e_1 is robust, degenerate and isolated. The example shows that the class of nondegenerate LCP's of size 2 is a proper subset of the class of robust LCP's.

The following example shows that the class of robust linear complementarity problems of size n is not open for general n.

Example 4.9. Take M, q and M_k ($k \in \mathbb{N}$) as in Remark 3.10 and let, for $\varepsilon > 0$,

$$M_k(\varepsilon) = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -k^{-1} & \varepsilon \\ 1 & k^{-1} & 0 \end{bmatrix}.$$

Then $S(q, M_k(\varepsilon)) = \{(1, 0, 1), (0, k, 0)\}$ for each $\varepsilon > 0$ and $k \in \mathbb{N}$. This shows that $(0, k, 1) \in S(q, M_k)$ is not a robust solution of (q, M_k), hence (q, M_k) is not robust.

Since $\lim_{k \to \infty}(q, M_k) = (q, M)$ and (q, M) is robust and even strongly stable, this implies that the class of robust or strongly stable linear complementarity problems of size 3 is not open.

Let OIND_n be the family of linear complementarity problems of size n, with at least one isolated and nondegenerate solution. OIND_n is a (proper) subfamily of the family of robust problems. This family is open as immediately follows from Theorem 3.8 and Lemma 3.7(iv). We conclude this section by giving an overview of the relations between all kinds of solutions which were discussed in this paper:

$$
\begin{tikzcd}
\boxed{\text{regular}} \arrow[d, Leftrightarrow] & \\
\boxed{\text{isolated nondegenerate}} \arrow[d, Rightarrow] & \\
\boxed{\text{strongly stable}} \arrow[d, Rightarrow] & \\
\boxed{\text{stable}} \arrow[r, Leftrightarrow] & \boxed{\text{robust isolated}} \arrow[d, Rightarrow] \\
& \boxed{\text{robust}}
\end{tikzcd}
$$

5. Robustness and nondegenerateness for a subclass of linear complementarity problems

By examples it was shown in the foregoing section that neither the family of nondegenerate LCP's of size n nor the family of robust LCP's is open for general n. If one considers the examples in 3.10 and 4.9 more carefully, then one can expect that the difficulties in these examples arise because a sequence $z(1), z(2), \ldots$ of solutions of a convergent sequence of LCP's may disappear to infinity, i.e. $\limsup_{k\to\infty}\|z(k)\|_\infty = \infty$. In this section we look at a subclass G_n, where this phenomenon does not arise. Here G_n is the family $\{(q, M) \in \mathbb{R}^n \times \mathbb{R}^{n\times n}\colon S(0, M) = \{0\}\}$. We note that the set of $n \times n$-matrices M with $S(0, M) = \{0\}$, coincides with the class $E^*(0)$, introduced by Garcia [5]. Hence, $G_n = \mathbb{R}^n \times E^*(0)$. Doverspike [4] notes that $M \in E^*(0)$ if and only if each $n \times n$-submatrix N of the $n \times 2n$-matrix $[I_n - M]$ for which the i-th column is either e_i or the i-th column of $-M$, generates a strictly pointed cone, i.e.,

$$\{x\colon Nx = 0, x \geqslant 0\} = \{0\}.$$

By Proposition 2.1 of Doverspike [4], the set of strictly pointed $n \times n$-matrices is open in $\mathbb{R}^{n\times n}$. Because there are a finite number of such matrices, N, it then follows that $E^*(0)$ and G_n are open sets.

Some interesting properties of linear complementarity problems in G_n are collected in the following proposition.

Proposition 5.1. *Let (q, M) be a linear complementarity problem of size n. Then*

(i) *$M \in E^*(0)$ iff $S(0, M)$ is bounded.*

(ii) *If $(q, M) \in G_n$, then $S(q, M)$ is bounded.*

(iii) *Let $(q(1), M(1)), q(2), M(2)), \ldots$ be a sequence of linear complementarity problems of size n converging to (q, M) and $z(1), z(2), \ldots$ a sequence with $z(k) \in S(q(k), M(k))$ for all $k \in \mathbb{N}$, and $\limsup_{k\to\infty} \|z(k)\|_\infty = \infty$. Then $(q, M) \notin G_n$.*

Proof. (i) follows from the fact that $z \in S(0, M)$ iff $\lambda z \in S(0, M)$ for all $\lambda \geqslant 0$.

(ii) Suppose that $S(q, M)$ is unbounded for (q, M). Then one of the finite number of polyhedral convex components is unbounded and contains a halfray of the form

$$\{w + \lambda z: \lambda \geqslant 0\}, \quad \text{where } z \neq 0.$$

Then it is straightforward (cf. Cottle [1, p. 61]) to show that $z \in S(0, M)$ and then $(q, M) \notin G_n$. This proves (ii).

(iii) Take a subsequence $z(t(1)), z(t(2)), \ldots$ of $z(1), z(2), \ldots$ such that sequence $y(t(1)), y(t(2)), \ldots$ with $y(t(k)) := \|z(t(k))\|_\infty^{-1} z(t(k))$ converges to an element y with $\|y\|_\infty = 1$. Then $y \geqslant 0$,

$$My = \lim_{k\to\infty} M(t(k))y(t(k)) \geqslant -\lim_{k\to\infty} \|z(t(k))\|_\infty^{-1} q(t(k)) = 0$$

and

$$\langle y, My\rangle = \lim_{k\to\infty} \langle y(t(k)), M(t(k))y(t(k)) + \|z(t(k))\|_\infty^{-1} q(t(k))\rangle = 0.$$

Hence $0 \neq y \in S(0, M)$. So $(q, M) \notin G_n$. □

Combining parts (i) and (ii) of the foregoing proposition, we find a new characterization of Garcia's class $E^*(0)$.

Corollary 5.2. *The solution set $S(q, M)$ of a linear complementarity problem (q, M) of size n is bounded (possibly empty) for all $q \in \mathbb{R}^n$ if and only if $M \in E^*(0)$.*

In the next two theorems we are interested in properties of the family $\mathrm{NDG}_n :=$ $\{(q, M) \in G_n : (q, M)$ is nondegenerate$\}$. For related results we refer to [2, 4, 8, 16, 17, 18].

Theorem 5.3. *The set* NDG_n *is open in* $\mathbb{R}^m \times \mathbb{R}^{n\times n}$.

Proof. Suppose that NDG_n is not an open subset of $\mathbb{R}^n \times \mathbb{R}^{n\times n}$. Then there exists a $(q, M) \in \mathrm{NDG}_n$ and a sequence $(q(1), M(1))$, $(q(2), M(2)), \ldots$ in $G_n - \mathrm{NDG}_n$, converging to (q, M). For large $k \in \mathbb{N}$, say for $k \geq K$, we can find a convex component $P(k)$ of $S(q(k), M(k))$ for which either $P(k)$ is not a one-point set or $P(k)$ consists of a degenerate solution. For each $k \geq K$, we take $\hat{z}(k) \in \operatorname{relint}(P(k))$. Then we may suppose that (a subsequence of) the sequence $\hat{z}(K), \hat{z}(K+1), \ldots$ converges, say to z, since otherwise we could find a subsequence of $\hat{z}(K), \hat{z}(K+1), \ldots$ disappearing to infinity, which is impossible by Proposition 5.1(iii). By Lemma 3.7, $z \in S(q, M)$. Since $(q, M) \in \mathrm{NDG}_n$, z is a nondegenerate and isolated solution. Hence, for large k, we have in view of Lemma 3.7(iv), that $\hat{z}(k)$ is a nondegenerate and isolated solution. This, however, contradicts with our assumptions about the set $P(k)$. So NDG_n must be open. $\square$

Theorem 5.4. (i) *The set* $\{(q, M) \in \mathrm{NDG}_n : |S(q, M)| = k\}$ *is open in* $\mathbb{R}^n \times \mathbb{R}^{n\times n}$ *for each* $k \in \mathbb{N}$.

(ii) *The number of solutions is locally constant on* NDG_n.

Proof. Since $\mathrm{NDG}_n = \bigcup_k \{(q, M) \in \mathrm{NDG}_n : |S(q, M)| = k\}$, the proof is complete if we can show that (i) is true. This, however, is an easy consequence of the following Lemma 5.5. $\square$

Note, that in view of Theorem 2.2, NDG_n is in fact a finite union of open sets, where the size of the solution set is constant.

In the following lemma we describe the behaviour of the cardinality of the solution set of problems in NDG_n.

Lemma 5.5. *Let* $(q, M), (q(1), M(1)), (q(2), M(2)), \ldots$ *be a sequence in* NDG_n *with* $(q, M) = \lim_{k\to\infty}(q(k), M(k))$. *Then there exists a* $K \in \mathbb{N}$ *such that* $|S(q, M)| = |S(q(k), M(k))|$ *for all* $k \geq K$.

Proof. Choose an $\varepsilon > 0$ such that

$$S(q, M) \cap B_\varepsilon(z) = \{z\} \quad \text{for all } z \in S(q, M).$$

(i) Firstly, we show that there exists an $L_1 \in \mathbb{N}$ such that, for each $k \geq L_1$,

$$S(q(k), M(k)) \subset \bigcup_{z\in S(q,M)} B_\varepsilon(z). \tag{5.1}$$

If such an L_1 should not exist, then one could find a sequence $z(k_1), z(k_2), \ldots$ such that, for each $j \in \mathbb{N}$,

(a) $z(k_j)$ is a solution of the problem $(q(k_j), M(k_j))$,

(b) $\{z(k_j)\} \cap B_\varepsilon(z) = \emptyset$ for all $z \in S(q, M)$.

Since $(q, M) \in G_n$, we may suppose, in view of Proposition 5.1(iii), that there is a convergent subsequence of $z(k_1), z(k_2), \ldots$ with limit, say $\bar{z}$. Then $\bar{z} \in S(q, M)$ by Lemma 3.7(ii) and $\{\bar{z}\} \cap B_\varepsilon(z) = \emptyset$ for all $z \in S(q, M)$ and this is impossible.

(ii) By the assumption that $(q, M) \in \mathrm{NDG}_n$, $S(q, M)$ is finite and each solution is isolated and nondegenerate. Therefore, by Remark 4.3, all solutions are strongly stable. This implies the existence of a $K \in \mathbb{N}$ and an $\varepsilon > 0$ such that for all $k \geqslant K$ and any $z \in S(q, M)$ we have

$$|S(q(k), M(k)) \cap B_\varepsilon(z)| = 1. \tag{5.2}$$

(iii) From (5.1) and (5.2) it follows that

$$|S(q, M)| = |S(q(k), M(k))| \quad \text{for all } k \geqslant K. \quad \square$$

Finally we deal with the class of robust linear complementarity problems (q, M) of fixed size with $M \in E^*(0)$. In Section 3 we have noticed that the multifunction S, which assigns to a linear complementarity problem its solution set, is closed. The next theorem concerns the upper and lower semicontinuity of this multifunction. We say that the multifunction S is *upper* [*lower*] *semicontinuous* in $(q, M) \in \mathbb{R}^n \times \mathbb{R}^{n \times n}$ if for each open set V with $V \supset S(q, M)$ [$V \cap S(q, M) \neq \emptyset$] there is a neighbourhood U of (q, M) such that

$$V \supset S(q', M') \; [V \cap S(q', M') \neq \emptyset] \quad \text{for any } (q', M') \in U.$$

Theorem 5.6. *Let $(q, M) \in G_n$ be a linear complementarity problem with a nonempty solution set. Then*

(i) *the multifunction S is upper semicontinuous in (q, M),*

(ii) *the multifunction S is lower semicontinuous in (q, M) if and only if (q, M) is robust.*

Proof. Since (ii) is obvious we only prove (i). Suppose that S is not upper semicontinuous in (q, M). Then we can find an $\varepsilon > 0$ and a sequence $(q(1), M(1))$, $(q(2), M(2)), \ldots$ in G_n converging to (q, M) such that for each $k \in \mathbb{N}$ there exists a $z(k) \in S(q(k), M(k))$ with $z(k) \notin B_\varepsilon(S(q, M))$. In view of Proposition 5.1(iii) we may suppose that the sequence $z(1), z(2), \ldots$ converges, say to z. Then $z \notin S(q, M)$, which contradicts the closedness of the multifunction S. $\square$

Remark 5.7. In Theorem 5.6 we have in fact proved that for a linear complementarity problem $(q, M) \in G_n$ with a nonempty solution set, (q, M) is robust if and only if S is both upper and lower semicontinuous in (q, M). If we provide the class C of compact subsets of $\mathbb{R}^n_+$ with the Hausdorff metric, then one can also prove that (q, M) is robust if and only if (q, M) is a point of continuity of the restriction of the map $S : G_n \to C$ to the set of linear complementarity problems with a non-empty solution set.

Acknowledgements

The authors are indebted to the referees for helpful comment and for suggestions, which led to shorter and simpler proofs of Lemma 2.3 and Theorem 4.7.

References

[1] R.W. Cottle, "Solution rays for a class of complementarity problems," *Mathematical Programming Study* 1 (1974) 59–70.

[2] R.W. Cottle, "Some recent developments in linear complementarity theory", in: R.W. Cottle, F. Giannessi and J-L. Lions, eds., *Variational inequalities and complementarity problems* (John Wiley, New York, 1980) pp. 97–104.

[3] R.W. Cottle and G.B. Dantzig, "Complementarity pivot theory of mathematical programming," *Linear Algebra and its Applications* 1 (1968) 103–125.

[4] R.D. Doverspike, "Some perturbation results for the linear complementarity problem," *Mathematical Programming* 23 (1982) 181–192.

[5] C.B. Garcia, "Some classes of matrices in linear complementarity theory," *Mathematical Programming* 5 (1974) 299–310.

[6] C.D. Ha, "Stability of the linear complementarity problem at a solution point," *Mathematical Programming* 31 (1985) 327–338.

[7] M.J.M. Jansen, "On the structure of the solution set of a linear complementarity problem," *Cahiers du C.E.R.O.* 25 (1983) 41–48.

[8] L.M. Kelly and L.T. Watson, "Erratum: Some perturbation theorems for Q-matrices," *SIAM Journal of Applied Mathematics* 34 (1978) 320–321.

[9] C.E. Lemke, "Bimatrix equilibrium points and mathematical programming," *Management Science* 11 (1965) 681–689.

[10] C.E. Lemke, "A survey of complementarity theory," in: R.W. Cottle, F. Giannessi and J-L. Lions, eds., *Variational inequalities and complementarity problems* (John Wiley, New York, 1980) pp. 213–239.

[11] O.L. Mangasarian, "Locally unique solutions of quadratic programs, linear and nonlinear complementarity problems," *Mathematical Programming* 19 (1980) 200–212.

[12] O.L. Mangasarian, "Characterization of bounded solutions of linear complementarity problems," *Mathematical Programming Study* 19 (1982) 153–166.

[13] K.G. Murty, "On the number of solutions to the complementarity problem and spanning properties of complementary cones," *Linear Algebra and its Applications* 5 (1972) 65–108.

[14] S.M. Robinson, "Strongly regular generalized equations," *Mathematics of Operations Research* 5 (1980) 43–62.

[15] R.T. Rockafellar, *Convex Analysis* (Princeton University Press, Princeton, NJ, 1970).

[16] A. Tamir, "The complementarity problem of mathematical programming," Ph.D. dissertation, Case Western Reserve University (Cleveland, OH, 1973).

[17] L.T. Watson, "A variational approach to the linear complementarity problem," Ph.D. dissertation, Department of Mathematics, University of Michigan (Ann Arbor, MI, 1974).

[18] L.T. Watson, "Some perturbation theorems for Q-matrices," *SIAM Journal of Applied Mathematics* 31 (1976) 379–384.